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HARMONIC ALMOST-COMPLEX STRUCTURES ON TWISTOR SPACES

BY

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ABSTRACT

We prove that the Atiyah–Hitchin–Singer [1] and Eells–Salamon [6] almost-complex structures on the negative twistor space of an oriented Riemannian four-manifold are harmonic in the sense of C. Wood [17, 18] if and only if the base manifold is, respectively, self-dual or self-dual and of constant scalar curvature. The stability of these almost-complex structures is also discussed.

1. Introduction

Let (N, h) be a connected even-dimensional Riemannian manifold. An almost-complex structure J on N is called almost-Hermitian, if it is h-orthogonal, i.e., $h(JX, JY) = h(X, Y)$ for $X, Y \in TN$. If (N, h) admits an almost-Hermitian structure, then it has many, and it is natural to seek for "reasonable" criteria that distinguish some of these structures. A natural way to obtain such criteria is to consider the almost-Hermitian structures on (N, h) as sections of its twistor bundle $\mathcal{T}$. Recall that $\mathcal{T}$ is the bundle over N whose fibre at a point $p \in N$ consists of all h-orthogonal complex structures on the tangent space T_pN. If the manifold N is oriented, the twistor space $\mathcal{T}$ has two connected components

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$\mathcal{T}_{\pm}$ whose sections are the almost-Hermitian structures compatible with $\pm$ the orientation of N.

E. Calabi and H. Gluck [3] have proposed to single out those almost-Hermitian structures J on (N, h), whose image $J(N)$ in $\mathcal{T}$ is of minimal volume with respect to the natural Riemannian metric $\tilde{h}$ on $\mathcal{T}$ induced by h and the standard metric of the fibre. Considering the 6-sphere S^6 in this context, they have proved that its canonical almost-Hermitian structure defined by means of the Cayley numbers can be characterized by that property.

Motivated by the harmonic maps theory, C. Wood [17, 18] has suggested to consider as "optimal" those almost-Hermitian structures J: $(N, h) \to (\mathcal{T}, \tilde{h})$, which are critical points of the energy functional under variations through sections of $\mathcal{T}$. In general, these critical points are not harmonic maps, but, by analogy, in [17, 18] they are referred to as "harmonic almost-complex structures". The Euler–Lagrange equation for a harmonic almost-complex structure J is [17, 18]

$$[J, \nabla^*\nabla J] = 0, \tag{1}$$

where $\nabla^*\nabla$ is the rough Laplacian of (N, h), so the Kähler structures are harmonic (in fact they are absolute minima of the energy functional). Moreover, several interesting examples of non-Kähler harmonic almost-complex structures J have been discussed in [17, 18] based on the observation [18, Theorem 2.8] that the Euler–Lagrange equation takes an apparently simple form in the case when J is integrable or $(1,2)$-symplectic ("quasi-Kähler" in the terminology of Gray-Hervella [9]).

The main purpose of this paper is to study the harmonicity of the Atiyah–Hitchin–Singer [1] and Eells–Salamon [6] almost-complex structures J_1 and J_2 on the negative twistor space $(\mathcal{Z}, h_t)$, $\mathcal{Z} = \mathcal{T}_-$, of an oriented Riemannian four-manifold, where $h_t, t > 0$, is the metric on $\mathcal{Z}$ obtained by rescaling $\tilde{h}$ on the fibres by t. Our main result is the following theorem, which may be considered as a variational interpretation of the self-duality condition:

THEOREM 1: *Let (M, g) be an oriented Riemannian 4-manifold and $(\mathcal{Z}, h_t)$ be its negative twistor space. Then:*
(i) *The Atiyah–Hitchin–Singer almost-complex structure J_1 on $(\mathcal{Z}, h_t)$ is harmonic, if and only if (M, g) is a self-dual manifold.*
(ii) *The Eells–Salamon almost-complex structure J_2 on $(\mathcal{Z}, h_t)$ is harmonic, if and only if (M, g) is a self-dual manifold with constant scalar curvature.*

To prove Theorem 1, we observe first, by means of the Weitzenböck formula, that the Euler–Lagrange equation (1) is equivalent to the condition that the

Laplacian of the Kähler form and its image under the curvature operator have equal [(2,0)+(0,2)]-parts. In the case of the twistor space $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$, we obtain explicit formulas for these 2-forms in terms of the curvature of the base manifold (M, g) and prove that the above condition is fulfilled for the horizontal vectors of $\mathcal{Z}$, if and only if the base manifold is self-dual. Finally we show that if (M, g) is self-dual, the Euler–Lagrange equation for J_1 is automatically satisfied, whereas, for J_2, it is equivalent to (M, g) being of constant scalar curvature.

In the last section of the paper we discuss the stability of the almost-complex structures J_1 and J_2.

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2. Preliminaries

Let M be a (connected) Riemannian manifold with metric g. Then g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by the formula

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2}[g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)].$$

The Riemannian connection of M determines a connection on the vector bundle $\Lambda^2 TM$ (both denoted by ∇) and the respective curvatures are related by

$$R(X, Y)(Z \wedge T) = R(X, Y)Z \wedge T + Z \wedge R(X, Y)T$$

for $X, Y, Z, T \in \chi(M)$; $\chi(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R we adopt the following definition: $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.) The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T)$$

for all $X, Y, Z, T \in \chi(M)$.

If M is oriented and of dimension four, the Hodge star operator defines a self-adjoint endomorphism $*$ of $\Lambda^2 TM$ with $*^2 = Id$. Hence

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM,$$

where $\Lambda^2_{\pm}TM$ are the subbundles of Λ^2TM corresponding to the (± 1)-eigenvectors of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$
\begin{aligned}
s_1 &= E_1 \wedge E_2 - E_3 \wedge E_4, & \bar{s}_1 &= E_1 \wedge E_2 + E_3 \wedge E_4,\\
s_2 &= E_1 \wedge E_3 - E_4 \wedge E_2, & \bar{s}_2 &= E_1 \wedge E_3 + E_4 \wedge E_2,\\
s_3 &= E_1 \wedge E_4 - E_2 \wedge E_3, & \bar{s}_3 &= E_1 \wedge E_4 + E_2 \wedge E_3.
\end{aligned} \tag{2}
$$

Then (s_1, s_2, s_3) (resp. $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$) is a local oriented orthonormal frame of Λ^2_-TM (resp. Λ^2_+TM). The matrix of $\mathcal{R}$ with respect to the frame $(\bar{s}_i, s_i)$ of Λ^2TM has the form

$$\mathcal{R} = \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix},$$

where the 3×3 matrices A and C are symmetric and have equal traces. Let $\mathcal{B}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be the endomorphisms of Λ^2TM with matrices

$$\mathcal{B} = \begin{bmatrix} 0 & B \\ {}^tB & 0 \end{bmatrix}, \quad \mathcal{W}_+ = \begin{bmatrix} A - \frac{s}{6}I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - \frac{s}{6}I \end{bmatrix},$$

where s is the scalar curvature and I is the unit 3×3 matrix. Then

$$\mathcal{R} = \frac{s}{6}Id + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_- \tag{3}$$

is the irreducible decomposition of $\mathcal{R}$ under the action of $SO(4)$ found by Singer and Thorpe [16]. Note that $\mathcal{B}$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ represent the traceless Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is called self-dual (anti-self-dual), if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The negative twistor space $\mathcal{Z}$ of M can be identified with the subbundle $\mathcal{Z}$ of Λ^2_-TM consisting of all unit vectors. The Riemannian connection ∇ of M gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. More precisely, let $\pi : \Lambda^2_-TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in \mathcal{Z}$ is $\mathcal{V}_\sigma = \mathrm{Ker}\pi_{*\sigma}$ ($T_\sigma\mathcal{Z}$ is always considered as a subspace of $T_\sigma(\Lambda^2_-TM)$). Note that $\mathcal{V}_\sigma$ consists of those vectors of $T_\sigma\mathcal{Z}$ which are tangent to the fibre $\mathcal{Z}_p = \pi^{-1}(p) \cap \mathcal{Z}$, $p = \pi(\sigma)$, of $\mathcal{Z}$ through the point σ. Since $\mathcal{Z}_p$ is the unit sphere in the vector space $\Lambda^2_-T_pM$, $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\Lambda^2_-T_pM$. Let s be a local section of $\mathcal{Z}$ such that $s(p) = \sigma$. Since s has constant length, $\nabla_X s \in \mathcal{V}_\sigma$ for all $X \in T_pM$. Given $X \in T_pM$, the vector $X^h_\sigma = s_*X - \nabla_X s \in T_\sigma\mathcal{Z}$ depends only on p and σ. By definition, the horizontal space at σ is $\mathcal{H}_\sigma = \{X^h_\sigma : X \in T_pM\}$. Note that the map $X \to X^h_\sigma$ is an isomorphism between T_pM and $\mathcal{H}_\sigma$ with inverse map $\pi_* \mid \mathcal{H}_\sigma$.

Let (U, x_1, x_2, x_3, x_4) be a local coordinate system of M and let (E_1, E_2, E_3, E_4) be an oriented orthonormal frame of TM on U. If (s_1, s_2, s_3) is the local frame of Λ^2_-TM defined by (2), then $\tilde{x}_i = x_i \circ \pi$, $y_j(\sigma) = g(\sigma, (s_j \circ \pi)(\sigma))$, $1 \leq i \leq 4$, $1 \leq j \leq 3$, are local coordinates of Λ^2_-TM on $\pi^{-1}(U)$. For each vector field

$$X = \sum_{i=1}^{4} X^i \frac{\partial}{\partial x_i}$$

on U the horizontal lift X^h of X on $\pi^{-1}(U)$ is given by

$$X^h = \sum_{i=1}^{4} (X^i \circ \pi) \frac{\partial}{\partial \tilde{x}_i} - \sum_{j,k=1}^{3} y_j g(\nabla_X s_j, s_k) \circ \pi \frac{\partial}{\partial y_k}. \tag{4}$$

Let $\sigma \in \mathcal{Z}$ and $\pi(\sigma) = p$. Using (4), and the standard identification $T_\sigma(\Lambda^2_- T_pM) \cong \Lambda^2_- T_pM$, one sees that

$$[X^h, Y^h]_\sigma = [X, Y]^h_\sigma + R_p(X \wedge Y)\sigma \tag{5}$$

for all (local) vector fields X, Y on M.

Each point $\sigma \in \mathcal{Z}$ defines a complex structure K_σ on T_pM by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y), \ \ X, Y \in T_pM. \tag{6}$$

Note that K_σ is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of K_σ.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_- T_pM$, $p \in M$. Then it is easily verified that

$$g(R(a)b, c) = -g(\mathcal{R}(a), b \times c)) \tag{7}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_- T_pM$ and

$$g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = -g(V, X \wedge Y) \tag{8}$$

for $V \in \mathcal{V}_\sigma$, $X, Y \in T_pM$.

Following [1] and [6], define two almost-complex structures J_1 and J_2 on $\mathcal{Z}$ by

$$J_n V = (-1)^n \sigma \times V \quad \text{for } V \in \mathcal{V}_\sigma,$$
$$J_n X^h_\sigma = (K_\sigma X)^h_\sigma \quad \text{for } X \in T_pM, p = \pi(\sigma).$$

It is well-known [1] that J_1 is integrable (i.e., comes from a complex structure), if and only if M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable [6].

Let h_t be the Riemannian metric on $\mathcal{Z}$ given by

$$h_t = \pi^* g + t g^v$$

where $t > 0$, g is the metric of M, and g^v is the restriction of the metric of $\Lambda^2 TM$ on the vertical distribution $\mathcal{V}$. Then π: $(Z, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres and the almost-complex structures J_1 and J_2 are compatible with the metrics h_t.

Now denote by D $(= D_t)$ the Levi-Civita connection of the metric h_t on $\mathcal{Z}$. Let X, Y be vector fields on M and V be a vertical vector field on $\mathcal{Z}$. Then it is easy to see that at any point $\sigma \in \mathcal{Z}$

$$(D_{X^h} Y^h)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2} R(X, Y)\sigma, \tag{9}$$

$$(D_V X^h)_\sigma = \mathcal{H}(D_{X^h} V)_\sigma = \frac{t}{2}(R(\sigma \times V)X)^h_\sigma. \tag{10}$$

Indeed, the first equality follows from (5) and the standard formula for the Levi-Civita connection in terms of inner products and Lie brackets. As to the second equality, let us note that $D_V X^h$ is perpendicular to any vertical vector field W, since $D_V W$ is a vertical vector field. Hence $D_V X^h$ is a horizontal vector field. On the other hand, $[V, X^h]$ is a vertical vector field, hence $D_V X^h = \mathcal{H} D_{X^h} V$. Then $h_t(D_V X^h, Y^h) = h_t(D_{X^h} V, Y^h) = -h_t(V, D_{X^h} Y^h)$ and (10) follows from (7), and (9).

3. Proof of Theorem 1

Let us note first that the Euler–Lagrange equation (1) for an almost-Hermitian structure J on a Riemannian manifold (N, h) can be written in the form

$$\nabla^* \nabla \Omega(A, B) = \nabla^* \nabla \Omega(JA, JB)$$

for every $A, B \in TN$ where Ω is the Kähler form of J. Further on we shall freely identify the 2-forms on N with the corresponding 2-vectors under the standard isomorphism $\Lambda^2 T^* N \cong \Lambda^2 TN$ induced by the metric h. By the Weitzenböck formula for 2-forms (cf., e.g., [5]) we have

$$\Delta\Omega(A, B) - \nabla^* \nabla \Omega(A, B) = \mathcal{R}(\Omega)(A, B) + Ric(A, JB) - Ric(JA, B),$$

where Δ is the Laplacian and Ric is the Ricci tensor. Therefore, J is harmonic if and only if

$$\Delta\Omega(A, B) - \Delta\Omega(JA, JB) = \mathcal{R}(\Omega)(A, B) - \mathcal{R}(\Omega)(JA, JB) \tag{11}$$

for every $A, B \in TN$.

Since the vertical spaces of the twistor space $\mathcal{Z}$ are 2-dimensional and J_n-invariant, both sides of (11) vanish on all vertical vectors. Therefore, it is enough to consider (11) only in the cases when A and B are horizontal vectors, or A is vertical and B is horizontal.

Denote by $\Omega_{n,t}(A,B) = h_t(J_nA, B), n = 1,2$, the Kähler form of the almost-Hermitian structure J_n on $(\mathcal{Z}, h_t)$.

LEMMA 1: *Let V be a vertical vector of $\mathcal{Z}$ at a point σ and $X, Y \in T_pM$, $p = \pi(\sigma)$. Then*

$$(12)\quad \Delta\Omega_{n,t}(X^h, Y^h)_\sigma = g\Big(\frac{4\sigma}{t} + 2(-1)^n\mathcal{R}(\sigma), X\wedge Y\Big) + tg(R(X\wedge Y)\sigma, R(\sigma)\sigma)$$

and

$$(13)\qquad \Delta\Omega_{n,t}(V, X^h)_\sigma = (-1)^{n+1}tg(\delta\mathcal{R}(X), V) - tg((\nabla_X\mathcal{R})(\sigma), \sigma\times V)$$

where Δ is the Laplacian of the metric h_t.

Proof: Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM near p such that for the local frame (s_1, s_2, s_3) of Λ^2_-TM defined by (2) we have $s_1(p) = \sigma$. Let $y_j(\tau) = g(\tau, (s_j\circ\pi)(\tau))$, $\tau\in\Lambda^2_-$, $1\le j\le 3$, and set

$$(14)\qquad U = (y_1^2+y_2^2)^{-1/2}\Big(-y_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_2}\Big).$$

Then

$$(15)\qquad J_1U = (y_1^2+y_2^2)^{-1/2}\Big(y_1y_3\frac{\partial}{\partial y_1} + y_2y_3\frac{\partial}{\partial y_2} - (1-y_3^2)\frac{\partial}{\partial y_3}\Big)$$

and $(\frac{1}{\sqrt{t}}U, \frac{1}{\sqrt{t}}J_1U)$ is a h_t-orthonormal frame of the vertical bundle $\mathcal{V}$ near the point σ such that

$$(16)\qquad (D_UU)_\sigma = (D_{J_1U}J_1U)_\sigma = 0.$$

First we shall prove that

$$(17)\qquad \delta d\Omega_{n,t}(X^h, Y^h)_\sigma = g\Big(\frac{4\sigma}{t} + 2(-1)^n\mathcal{R}(\sigma), X\wedge Y\Big).$$

To do this we shall use the fact (cf. [15]) that the differential of $\Omega_{n,t}$ is given by

$$d\Omega_{n,t}(X^h, Y^h, V) = g(2V + (-1)^nt\mathcal{R}(V), X\wedge Y);$$
$$(18)\qquad d\Omega_{n,t}(A,B,C) = 0$$

if A, B, C are horizontal vectors, or at least two of them are vertical.

This formula, together with (9), (10), (7), and (16), implies

$$\begin{aligned}
\delta d\Omega_{n,t}(X^h,Y^h)_\sigma =& -\sum_{i=1}^{4}(D_{E_i^h}d\Omega_{n,t})(E_i^h,X^h,Y^h)_\sigma\\
&-\frac{1}{t}(D_U d\Omega_{n,t})(U,X^h,Y^h)_\sigma-\frac{1}{t}(D_{J_1U}d\Omega_{n,t})(J_1U,X^h,Y^h)_\sigma\\
=&-\frac{1}{t}U_\sigma(d\Omega_{n,t}(U,X^h,Y^h))-\frac{1}{t}(J_1U)_\sigma(d\Omega_{n,t}(J_1U,X^h,Y^h))\\
=&-\frac{2}{t}\Big(\frac{\partial}{\partial y_2}\Big)_\sigma(g(U,(X\wedge Y)\circ\pi))+(-1)^{n+1}\Big(\frac{\partial}{\partial y_2}\Big)_\sigma(g(U,\mathcal{R}(X\wedge Y)\circ\pi))\\
&-\frac{2}{t}\Big(\frac{\partial}{\partial y_3}\Big)_\sigma(g(J_1U,(X\wedge Y)\circ\pi))-(-1)^{n+1}\Big(\frac{\partial}{\partial y_3}\Big)_\sigma(g(J_1U,\mathcal{R}(X\wedge Y)\circ\pi))\\
=&\frac{4}{t}g(s_1,X\wedge Y)+2(-1)^n g(\mathcal{R}(s_1),X\wedge Y),
\end{aligned}$$

in view of (14) and (15).

To compute $d\delta\Omega_{n,t}(X^h,Y^h)$ we need the following formula (cf. [15]):

$$\delta\Omega_{n,t}(E)=-tg(\mathcal{V}E,R(\sigma)\sigma)\quad\text{for any } E\in T_\sigma\mathcal{Z}. \tag{19}$$

Therefore, by (5),

$$d\delta\Omega_{n,t}(X^h,Y^h)=-\delta\Omega_{n,t}([X^h,Y^h]_\sigma)=tg(R(X\wedge Y)\sigma,R(\sigma)\sigma), \tag{20}$$

and then (12) follows from (17) and (20).

Now let s be a local section of $\mathcal{Z}$ such that $s(p)=\sigma$ and $\nabla s|_p=0$. If V is a vertical vector field on $\mathcal{Z}$ and X is a vector field on M, it follows easily from (4) that

$$[X^h,V]_\sigma=\nabla_{X_p}(V\circ s), \tag{21}$$

where $V\circ s$ is considered as a section of Λ^2_-TM. Since D_VX^h is a horizontal vector field (cf. (10)), we have

$$\mathcal{V}D_{X^h}V=\nabla_{X_p}(V\circ s). \tag{22}$$

Then formulas (18) and (22) imply that

$$\begin{aligned}
\delta d\Omega_{n,t}(V,X^h)_\sigma =& -s_{*p}E_i(d\Omega_{n,t}(E_i^h,V,X^h))+d\Omega_{n,t}((\nabla_{E_i}E_i)^h,V,X^h)_\sigma\\
&+d\Omega_{n,t}(E_i^h,\nabla_X(V\circ s),X^h)_\sigma+d\Omega_{n,t}(E_i^h,V,(\nabla_{E_i}X)^h)_\sigma\\
=&(-1)^{n+1}tg(\delta\mathcal{R}(X),V).
\end{aligned}\tag{23}$$

On the other hand, by (19) and (21), we have

$$\begin{aligned} d\delta\Omega_{n,t}(V,X^h)_\sigma &= -s_{*p}X(\delta\Omega_{n,t}(V)) - \delta\Omega_{n,t}([V,X^h]_\sigma) \\ &= tX(g(V\circ s),R(s)s)) - tg(\nabla_X(V\circ s),R(\sigma)\sigma) \end{aligned}$$

and using (7) we obtain

$$d\delta\Omega_{n,t}(V,X^h)_\sigma = -tg((\nabla_X\mathcal{R})(\sigma),\sigma\times V). \tag{24}$$

Thus, (13) follows from (23) and (24). ∎

Let R_t be the curvature tensor of $(\mathcal{Z},h_t)$. An explicit formula for the sectional curvature of $(\mathcal{Z},h_t)$ in terms of the curvature of (M,g) was obtained in [4]. Using this formula and the well-known expression of the Riemannian curvature tensor by means of sectional curvatures (cf., e.g., [10]), one gets easily the following lemma.

LEMMA 2: *Let V,W be vertical vectors of $\mathcal{Z}$ at a point σ and $X,Y\in T_pM$, $p=\pi(\sigma)$. Then*

$$\begin{aligned} \mathcal{R}_t(\Omega_{n,t})(X^h,Y^h)_\sigma =& 2[1+(-1)^{n+1}]g(\mathcal{R}(\sigma),X\wedge Y) - tg(R(X\wedge Y)\sigma,R(\sigma)\sigma) \\ &- \frac{t}{2}Trace\{Z\to g(R(X\wedge Z)\sigma,R(Y\wedge K_\sigma Z)\sigma)\} \\ &- \frac{t}{2}(-1)^nTrace\{\mathcal{V}_\sigma\ni\tau\to g(R(\tau)X,R(\sigma\times\tau)Y)\}, \end{aligned} \tag{25}$$

where the latter trace is taken with respect to the metric g on $\mathcal{V}_\sigma$,

$$\mathcal{R}_t(\Omega_{n,t})(V,X^h)_\sigma = tg((\nabla_X\mathcal{R})(\sigma),\sigma\times V)p \tag{26}$$

and

$$\begin{aligned} \mathcal{R}_t(\Omega_{n,t})(V,W)_\sigma =& 2[(-1)^{n+1}+tg(\mathcal{R}(\sigma),\sigma)]g(V,\sigma\times W) \\ &+ \frac{t^2}{2}Trace\{Z\to g(R(\sigma\times V)K_\sigma Z,R(\sigma\times W)Z)\}. \end{aligned}$$

Now we are ready to prove the theorem. According to (11), and Lemmas 1 and 2, the almost complex structure J_n is harmonic if and only if the following two conditions are satisfied:

$$\begin{aligned} 4g(\mathcal{R}(\sigma),X\wedge Y-K_\sigma X\wedge K_\sigma Y) =& \\ & tTrace\{Z\to g(R(X\wedge Z)\sigma,R(Y\wedge K_\sigma Z)\sigma) \\ &\qquad - g(R(K_\sigma X\wedge Z)\sigma,R(K_\sigma Y\wedge K_\sigma Z)\sigma)\} \\ &+ t(-1)^nTrace\{\mathcal{V}_\sigma\ni\tau\to g(R(\tau)X,R(\sigma\times\tau)Y) \\ &\qquad - g(R(\tau)K_\sigma X,R(\sigma\times\tau)K_\sigma Y)\} \end{aligned} \tag{27}$$

and

$$g(\delta\mathcal{R}(K_\sigma X), \sigma \times V) = (-1)^n g(\delta\mathcal{R}(X), V) \tag{28}$$

for every $\sigma \in \mathcal{Z}$, $V \in \mathcal{V}_\sigma$ and $X, Y \in T_{\pi(\sigma)}M$.

We shall prove that condition (27) is equivalent to (M, g) being a self-dual manifold. For that purpose first we show that the identity (27) can be expressed in terms of the self-adjoint operator

$$\mathcal{P} = \frac{s}{6} Id + \mathcal{W}_-.$$

Note that $X \wedge Y - K_\sigma X \wedge K_\sigma Y \in \mathcal{V}_\sigma$ for every $\sigma \in \mathcal{Z}$ and $X, Y \in T_{\pi(\sigma)}M$, which implies that

$$R(\tau)K_\sigma X = K_\sigma R(\tau)X + g(\mathcal{P}(\sigma \times \tau), \tau)K_\tau X - g(\mathcal{P}(\tau), \tau)K_{\sigma\times\tau}X$$

for any $\tau \in \mathcal{V}_\sigma, |\tau| = 1$. Let us note also that $K_\sigma \circ K_\tau = -K_{\sigma\times\tau}$. Using these facts it is easy to see that condition (27) is equivalent to the identity

$$\begin{aligned} &t(1+(-1)^{n+1})g(\mathcal{P}(\tau), \sigma \times \tau)g(\mathcal{P}(\tau), \sigma) = \\ &\qquad g(\mathcal{P}(\sigma), \sigma \times \tau)[4 + t(-1)^{n+1}g(\mathcal{P}(\tau), \tau) - tg(\mathcal{P}(\sigma \times \tau, \sigma \times \tau)] \end{aligned} \tag{29}$$

for every $\sigma \in \mathcal{Z}$, $\tau \in \mathcal{V}_\sigma$, $|\tau| = 1$. Replacing (σ, τ) consecutively by $(\tau, \sigma \times \tau)$ and by $(\sigma, \sigma \times \tau)$ in (29), and then adding the identities obtained, we get

$$g(\mathcal{P}(\sigma), \tau)[g(\mathcal{P}(\sigma), \sigma) - g(\mathcal{P}(\tau), \tau)] = 0, \quad \sigma \in \mathcal{Z},\ \tau \in \mathcal{V}_\sigma,\ |\tau| = 1. \tag{30}$$

Replacing the pair (σ, τ) by $(\frac{3\sigma+4\tau}{5}, \frac{4\sigma-3\tau}{5})$ in (30) and using again this identity, we obtain

$$[g(\mathcal{P}(\sigma), \sigma) - g(\mathcal{P}(\tau), \tau)]^2 = 4[g(\mathcal{P}(\sigma), \tau)]^2,$$

which, together with (30), gives

$$g(\mathcal{P}(\sigma), \sigma) = g(\mathcal{P}(\tau), \tau),\ g(\mathcal{P}(\sigma), \tau) = 0.$$

Thus

$$g(\mathcal{W}_-(\sigma), \sigma) = g(\mathcal{W}_-(\tau), \tau) \quad \text{and} \quad g(\mathcal{W}_-(\sigma), \tau) = 0$$

for every $\sigma, \tau \in \Lambda^2_- T_pM$, $\sigma \perp \tau$, $|\sigma| = |\tau| = 1, p \in M$. Since $Trace\mathcal{W}_- = 0$, this implies $\mathcal{W}_- = 0$.

Conversely, it is obvious that (29) is satisfied, if $\mathcal{W}_- = 0$.

To analyse condition (28) we recall that $\delta\mathcal{R} = 2\delta\mathcal{B}\,(= -dRic)$ (cf., e.g., [2]), so it follows from (3) that

$$\delta\mathcal{R}(X) = -\frac{1}{3}\operatorname{grad} s \wedge X + 2\delta\mathcal{W}(X),\ \ X \in TM.$$

Suppose $\mathcal{W}_- = 0$. Since $\delta\mathcal{W}_+(X) \in \Lambda^2_+TM$, we have

$$g(\delta\mathcal{R}(X), V) = \frac{1}{3}g(X \wedge \operatorname{grad} s, V)$$

for any $V \in \Lambda^2_-TM$. The latter formula and (8) imply that condition (28) is equivalent (for self-dual manifolds) to the identity

$$g(V, X \wedge \operatorname{grad} s) = (-1)^{n+1}g(V, X \wedge \operatorname{grad} s).$$

Obviously, this identity is satisfied if $n = 1$; for $n = 2$ it holds, if and only if the scalar curvature s is constant. ∎

4. Remarks on the stability of the almost-complex structures J_1 and J_2

A harmonic almost-complex structure J on a Riemannian manifold (N, h) is called stable if the second variation of the energy functional is non-negative for all compactly supported deformations of J through sections of the twistor space of (N, h). In [18], C. Wood has given various sufficient conditions for stability of a harmonic almost-complex structure. In particular, it follows from [18, Lemma 4.3 and the proof of Theorem 2.8(4)] that if a harmonic almost-complex structure J is integrable and cosymplectic (i.e., $\delta J = 0$) (resp., $(1, 2)$-symplectic), then it is stable provided the image $\mathcal{R}(\Omega)$ of the Kähler form Ω under the curvature operator $\mathcal{R}$ is a non-negative (resp., non-positive) $(1, 1)$-form. Moreover, under these hypotheses, J is a local minimum of the energy functional on any compactly supported variations of J through sections of the twistor space.

Consider the Atiyah–Hitchin–Singer almost-complex structure J_1 on the twistor space $(\mathcal{Z}, h_t)$ of an oriented Riemannian four-manifold (M, g). By Theorem 1, J_1 is harmonic iff (M, g) is self-dual, in which case J_1 is integrable [1] and cosymplectic (see (19)). Next we shall discuss the non-negativity of the $(1, 1)$-form $\mathcal{R}_t(\Omega_{1,t})$ on the twistor space of a self-dual manifold. It is easy to see that, in the notations of Lemma 2, we have

$$\begin{aligned} \mathcal{R}_t(\Omega_{1,t})(X^h, J_1X^h)_\sigma &= 4g(\mathcal{R}(\sigma), X \wedge K_\sigma X) \\ &- \frac{t}{2}[||R(\tau)X||^2 + ||R(\sigma \times \tau)X||^2 - 2g(R(\tau)X, K_\sigma R(\sigma \times \tau)X)], \end{aligned} \tag{31}$$

where $\tau \in \mathcal{V}_\sigma$ is an arbitrary g^v-unit vector,

$$\mathcal{R}_t(\Omega_{1,t})(V, X^h)_\sigma = 0 \tag{32}$$

and

$$\mathcal{R}_t(\Omega_{1,t})(V, J_1V)_\sigma = 2\Big[1 + \frac{ts}{6} - \Big(\frac{ts}{12}\Big)^2\Big] g^v(V,V). \tag{33}$$

Suppose that the $(1,1)$-form $\mathcal{R}_t(\Omega_{1,t})$ is non-negative. Then it follows from (31) and the Cauchy–Schwartz inequality that $g(\mathcal{R}(\sigma), X \wedge K_\sigma X) \geq 0$ for every $\sigma \in \mathcal{Z}$ and $X \in T_{\pi(\sigma)}M$ which is equivalent to the Ricci operator $\mathcal{R}ic = \frac{5}{6}sId + \mathcal{B}$ being non-negative. In particular, $s \geq 0$ and next we shall consider only the cases when $s = 0$, or $s > 0$ everywhere on M. If $s \equiv 0$, then the self-adjoint operator $\mathcal{B}$ is non-negative, and so $\mathcal{B} = 0$, since $Trace\mathcal{B} = 0$. Thus M is a self-dual Ricci flat manifold and, by a result of Hitchin [11], if it is compact, then either M is flat or its universal covering is a $K3$-surface with a Calabi–Yau metric. If $s > 0$ everywhere on M and M is compact, then by a result of P. Gauduchon [8] (see also [14]) either M is diffeomorphic to the connected sum $n\mathbb{CP}^2$, $0 \leq n \leq 3$ ($n\mathbb{CP}^2 = S^4$ for $n = 0$) or its universal covering is the Riemannian product $\mathbb{R}\times S^3$. Moreover, if the Ricci operator $\mathcal{R}ic$ is positive, then (M, g) is conformally equivalent to S^4 or $\mathbb{CP}^2$ with their standard metrics.

The aforementioned results lead us to consider the question, whether the $(1,1)$-form $\mathcal{R}_t(\Omega_{1,t})$ is non-negative in the following cases:

(a) *(M, g) is a self-dual Einstein manifold with non-negative scalar curvature.*

In this case, by (31), (32) and (33), $\mathcal{R}_t(\Omega_{1,t})$ is non-negative, if and only if $0 \leq ts \leq 12(1+\sqrt{2})$. Thus, if $s = 0$, the Atiyah–Hitchin–Singer complex structure J_1 is stable harmonic with respect to any metric h_t, $t > 0$, on the twistor space $\mathcal{Z}$; if $s > 0$, then J_1 is stable with respect to h_t provided $0 < t < \frac{12}{s}(1 + \sqrt{2})$. In both cases J_1 is a local energy-minimizer.

By a result of Hitchin [12] and Friedrich–Kurke [7], the only compact self-dual Einstein manifolds with positive scalar curvature are S^4 and $\mathbb{CP}^2$ with their standard metrics. The respective twistor spaces $(\mathcal{Z}, J_1)$ are the complex projective space $\mathbb{CP}^3$ and the complex flag manifold $F_{1,2}$, and the metrics h_t for $t = 12/s$ coincide with their standard Kähler metrics. So it seems interesting to find all $t > 0$ for which the complex structures of $\mathbb{CP}^3$ and $F_{1,2}$ are stable harmonic with respect to the metric h_t.

(b) *The universal covering of (M, g) is $\mathbb{R} \times S^3$ with the product metric.*

We may assume, without loss of generality, that $M = \mathbb{R} \times S^3$, where S^3 is endowed with the round metric of scalar curvature $s = 6$. Let $\sigma \in \mathcal{Z}$ and

$\pi(\sigma) = (r,a)$. For $X \in T_{\pi(\sigma)}(\mathbb{R}\times S^3)$, denote by X' the projection of X on the space $T_aS^3 \cap K_\sigma T_aS^3$. Then it is easy to compute by means of (31) that

$$\mathcal{R}_t(\Omega_{1,t})(X^h, J_1X^h) = 2\Big(2-\frac{t}{4}\Big)||X'||^2.$$

Therefore, in view of (32) and (33), $\mathcal{R}_t(\Omega_{1,t})$ is non-negative, if and only if $0 < t \leq 2(1+\sqrt{2})$, so for these values of t, J_1 is stable harmonic with respect to the metric h_t (in fact, a local minimum of the energy functional).

(c) *M is diffeomorphic to $2\mathbb{CP}^2$ or $3\mathbb{CP}^2$.*

In [13], C. LeBrun has associated a conformal class $C_{p_1,\dots,p_n}$ of self-dual metrics on $n\mathbb{CP}^2$ to any configuration of points $\{p_1,\dots,p_n\}$ in the hyperbolic 3-space. Each of these classes has a representative of positive scalar curvature [13]. Moreover, by a result of LeBrun–Nayatani–Nitta [14], every conformal class C_{p_1,p_2} on $2\mathbb{CP}^2$ contains a metric with strongly positive Ricci curvature and non-negative Ricci operator $\mathcal{R}ic$. It is not known to us, however, whether $2\mathbb{CP}^2$ and $3\mathbb{CP}^2$ admit self-dual metrics for which the $(1,1)$-forms $\mathcal{R}_t(\Omega_{1,t})$ on their twistor spaces are non-negative.

As is well-known, the Eells–Salamon almost complex structure J_2 is never integrable and it is $(1,2)$-symplectic with respect to h_t, if and only if the base manifold (M,g) is Einstein and self-dual. Moreover, if such a manifold is compact and $st = 6$, then (M,g) is isometric to S^4 or $\mathbb{CP}^2$ ([12, 7]) and J_2 is a nearly-Kähler structure on $\mathbb{CP}^3$ or $F_{1,2}$ such that $\mathcal{R}_t(\Omega_{2,t}) = \frac{s}{12}\Omega_{2,t}$. Having in mind that the standard almost-complex structure of S^6 has similar properties and is unstable harmonic [17], it is tempting to ask whether the same holds for the Eells–Salamon almost complex structure J_2 on $\mathbb{CP}^3$ and $F_{1,2}$.

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